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Oscillatory Phase of Inflaton and Power-Law Expansion in Bianchi Type-I Universe

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A homogeneous massive scalar field, minimally coupled to the spatially homogeneous and anisotropic background metric, in the semiclassical theory of gravity is examined. In the oscillatory phase of inflaton, the approximate leading solution to the semiclassical Einstein equation for the Bianchi type-I universe shows, each scale factor in each direction obeys $t^{2/3}$ power-law expansion. Further noted that the evolution of scale factors are mutually correlated.

KEY WORDS: anisotropic; cosmology; inflaton; power-law expansion.

1. INTRODUCTION

Although the present universe in its overall structure seems to be spatially homogeneous and isotropic, there are reasons to believe that it has not been so in all its evolution and that inhomogeneities and anisotropies might have played an important role in the early universe (Misner, 1969a,b). The isotropic model is adequate enough for the description of the later stages of evolution of the universe but this does not mean that the model is equally suitable for the description of early stages of the evolution, especially near the singularity (Landau and Lifshitz, 1979). The most general solutions of the problem of gravitational collapse turn to be locally anisotropic near the singularity (Belinski *et al.*, 1971; Heckman and Schucking, 1962; Thorne, 1967). Cosmological solutions of the Einstein general relativity are known in which the expansion be anisotropic at first, near the singularity, and later the expansion became isotropic. To avoid postulating specific initial conditions, as well as, the existence of particle horizon in isotropic models, attempts have been made through the study of inhomogeneous anisotropic universe. Among the anisotropic cosmological models the Bianchi type-I universe is the simplest one. In this model the metric is considered as spatially homogeneous and possibly anisotropic. In contrast to the Friedmann–Robertson–Walker (FRW)

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metric, the Bianchi type-I metric has a different scale factor in each direction. Therefore the expansion in this model can be visualized as anisotropic expansion. Interests in such models have been received much attention (Belinski *et al.*, 1971; Heckman and Schucking, 1962; Hu and Parker, 1978; Misner, 1969a,b; Throne, 1967; Zel'dovich and Starobinsky, 1971). Huang has considered the fate of symmetry in a Bianchi type-I cosmology using adiabatic approximation for massless field with arbitrary coupling to gravity (Huang, 1990). Futamas has studied the effective potential in the Bianchi type-I cosmology (Futamas, 1984). Berkin has examined the effective potential in the Bianchi type-I universe, for scalar field having arbitrary mass and coupling to gravity (Berkin, 1992). The first order phase transitions in the Bianchi type-I cosmology in the early universe also studied recently (Minu and Kuriakose, 2000).

Anisotropic models of the universe that become isotropic during evolution have been repeatedly considered (Belinski and Khalatnikov, 1972). These motivate the study of an anisotropic background cosmological model with the scalar field possess the advantage of the FRW model (Folomeev and Gurovich, 2000). These studies show, the Bianchi type-I cosmological model may be quite useful in the study of the early universe problems. The present work is to study a homogeneous massive scalar field (inflaton) in the Bianchi type-I universe in semiclassical theory of gravity, and hence to obtain an approximate leading solution to the semiclassical Einstein equation in the oscillatory phase of the inflaton, after inflation, and examine whether the scale factors in the Bianchi type-I cosmology follow the same power-law expansion as that of isotropic model in semiclassical theory.

2. INFLATON IN SEMICLASSICAL THEORY OF GRAVITY

Most of the cosmological models are based on the classical gravity of the Friedmann equation and scalar field equation on the FRW universe. To study the scalar field and the Friedmann equation at deeper level, both background metric and the field are to be treated quantum mechanically. Because a consistent quantum theory of gravity is not available, in most of the cosmological models, the background metric is considered as classical and matter field as quantum mechanical. Such an approximation of the Einstein equation is known as semiclassical approximation.

In semiclassical theory of gravity, the Einstein equation takes the following form (with $\hbar = c = k_B = 1$ and $G = \frac{1}{m_p^2}$):

$$G_{\mu\nu} = \frac{8\pi}{m_p^2}\langle \hat{T}_{\mu\nu}\rangle, \tag{1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and $\langle \hat{T}_{\mu\nu}\rangle$ is the expectation value of the energy–momentum tensor for a matter field in a suitable quantum state under consideration. In the above equation the quantum field, represented by a scalar field, ϕ, is governed by the time dependent Schrodinger equation:

$$i\frac{\partial}{\partial t}\Psi(\phi, t) = \hat{\mathcal{H}}_m(\phi, t)\Psi(\phi, t). \tag{2}$$

Consider a minimally coupled massive inflaton in the spatially flat Friedmann–Robertson–Walker metric:

$$ds^2 = -dt^2 + \mathcal{R}^2(t)(dx^2 + dy^2 + dz^2), \tag{3}$$

where $\mathcal{R}(t)$ is the scale factor representing the size of the universe. The purely temporal component of the classical gravity is the classical Einstein equation:

$$\left(\frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)}\right)^2 = \frac{8\pi}{m_p^2}\frac{T_{00}}{\mathcal{R}^3(t)}, \tag{4}$$

where

$$T_{00} = \mathcal{R}^3(t)\left(\frac{\dot{\phi}^2(t)}{2} + m^2\frac{\phi^2(t)}{2}\right), \tag{5}$$

is the energy density of the inflaton. The classical equation of motion for the inflaton is given by

$$\ddot{\phi}(t) + \left(\frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)}\right)\dot{\phi}(t) + m^2\phi(t) = 0. \tag{6}$$

In cosmological context, the classical Einstein equation means that the Hubble constant ($H = \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)}$) is determined by the energy density of the dynamically evolving homogeneous massive scalar field described by the classical equation of motion.

3. POWER-LAW EXPANSION IN BIANCHI TYPE-I UNIVERSE

The inflaton can be studied in the Bianchi type-I cosmology by using the basic equations discussed in the previous section. In the Bianchi type-I universe, a spatially homogeneous and anisotropic metric is given by

$$ds^2 = -dt^2 + \sum_{i=1}^{3}\mathcal{R}_i^2(t)\,dx_i^2, \tag{7}$$

where $\mathcal{R}_1(t)$, $\mathcal{R}_2(t)$, and $\mathcal{R}_3(t)$ are the scale factors in x, y, and z directions, respectively, which are representing the size of the universe in their respective direction. The Bianchi type-I model is an anisotropic generalization of the FRW model with the Euclidean spatial geometry. The three scale factors $\mathcal{R}_1(t)$, $\mathcal{R}_2(t)$, and $\mathcal{R}_2(t)$ are determined via the Einstein equation.

In the background metric (7), consider a homogeneous massive scalar field, minimally coupled to gravity, satisfying

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2)\phi = 0, \tag{8}$$

where ∇_μ is the covariant derivative and $\mu = 0, 1, 2, 3$; $g^{\mu\nu}$ is the reciprocal of the metric. The Lagrangian density for the scalar field, ϕ, is given by

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}(g^{\alpha\mu}\partial_\alpha\phi\partial_\mu\phi + m^2\phi^2), \tag{9}$$

where g is the determinant of the metric $g_{\mu\nu}$. For the metric (7), (8) becomes

$$\ddot{\phi}(t) + \sum_{i=1}^{3}\left(\frac{\dot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\right)\dot{\phi}(t) + m^2\phi(t) = 0. \tag{10}$$

In the present context Eq. (10) is the classical equation of motion for the scalar field for the metric (7).

The scalar field can be quantized, by canonical quantization rules, in a way consistent with the equation of motion, by defining the momentum conjugate to ϕ, as

$$\pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}. \tag{11}$$

The massive minimal inflaton in the Bianchi type-I cosmological model can be described by a time dependent harmonic oscillator, and can be obtained by using Eqs. (9) and (11) in following relation

$$\mathcal{H} = \pi_\phi\dot{\phi} - \mathcal{L}. \tag{12}$$

Therefore the Hamiltonian is obtained as

$$\mathcal{H}_m = \frac{1}{2\mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)}\pi_\phi^2 + \frac{m^2\mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)}{2}\phi^2(t) \tag{13}$$

For the metric (7), the purely temporal component of the classical gravity is now the classical Einstein equation:

$$\frac{\dot{\mathcal{R}}_1(t)}{\mathcal{R}_1(t)}\frac{\dot{\mathcal{R}}_2(t)}{\mathcal{R}_2(t)} + \frac{\dot{\mathcal{R}}_2(t)}{\mathcal{R}_2(t)}\frac{\dot{\mathcal{R}}_3(t)}{\mathcal{R}_3(t)} + \frac{\dot{\mathcal{R}}_1(t)}{\mathcal{R}_1(t)}\frac{\dot{\mathcal{R}}_3(t)}{\mathcal{R}_3(t)} = \frac{8\pi}{m_p^2}\frac{T_{00}}{\mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)}, \tag{14}$$

The eigenstate of the Hamiltonian can be constructed by using the annihilation and creation operators in the following manner:

$$\begin{aligned}\hat{a}(t) &= \phi^*(t)\hat{\pi}_\phi - \mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)\dot{\phi}^*(t)\hat{\phi}\\ \hat{a}^\dagger(t) &= \phi(t)\hat{\pi}_\phi - \mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)\dot{\phi}(t)\hat{\phi}\end{aligned} \tag{15}$$

Therefore the Fock space of the Hamiltonian is

$$\hat{a}^{\dagger}(t)\hat{a}(t)|\mathcal{N}, \phi, t\rangle = \mathcal{N}|\mathcal{N}, \phi, t\rangle \tag{16}$$

In the present context the semiclassical theory of Einstein equation takes the following form:

$$\frac{\dot{\mathcal{R}}_1(t)}{\mathcal{R}_1(t)}\frac{\dot{\mathcal{R}}_2(t)}{\mathcal{R}_2(t)} + \frac{\dot{\mathcal{R}}_2(t)}{\mathcal{R}_2(t)}\frac{\dot{\mathcal{R}}_3(t)}{\mathcal{R}_3(t)} + \frac{\dot{\mathcal{R}}_1(t)}{\mathcal{R}_1(t)}\frac{\dot{\mathcal{R}}_3(t)}{\mathcal{R}_3(t)} = \frac{8\pi}{m_p^2\mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)}\langle\hat{\mathcal{H}}\rangle \tag{17}$$

where $\mathcal{H}$ is given by Eq. (13).

The expectation value of $\mathcal{H}$ can be computed in number state $|\mathcal{N}, \phi, t\rangle$ by using (15) and (16) in (13) and apply the result in (17), we obtain the semiclassical Einstein equation as

$$\frac{\dot{\mathcal{R}}_1(t)}{\mathcal{R}_1(t)}\frac{\dot{\mathcal{R}}_2(t)}{\mathcal{R}_2(t)} + \frac{\dot{\mathcal{R}}_2(t)}{\mathcal{R}_2(t)}\frac{\dot{\mathcal{R}}_3(t)}{\mathcal{R}_3(t)} + \frac{\dot{\mathcal{R}}_1(t)}{\mathcal{R}_1(t)}\frac{\dot{\mathcal{R}}_3(t)}{\mathcal{R}_3(t)} = \frac{8\pi}{m_p^2}\left\{\left(\mathcal{N} + \frac{1}{2}\right)(\dot{\phi}^*\dot{\phi} + m^2\phi^*\phi)\right\} \tag{18}$$

In the above equations, ϕ and ϕ^* satisfy (6) and the boundary condition

$$\mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)(\dot{\phi}^*(t)\phi(t) - \phi^*(t)\dot{\phi}(t)) = i. \tag{19}$$

Transform the solution in the following form:

$$\phi(t) = \frac{1}{\sqrt{(\mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t))}}\eta(t) \tag{20}$$

Therefore (10) becomes

$$\ddot{\eta}(t) + \left(m^2 + \frac{1}{4}\sum_{i=1}^{3}\left(\frac{\dot{\mathcal{R}}_i}{\mathcal{R}_i}\right)^2 - \frac{1}{2}\sum_{i\neq j=1}^{3}\left(\frac{\dot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\frac{\dot{\mathcal{R}}_j(t)}{\mathcal{R}_j(t)}\right) - \frac{1}{2}\sum_{i=1}^{3}\frac{\ddot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\right)\eta(t) = 0. \tag{21}$$

Next, concentrating on the oscillatory phase of the inflaton after inflation in the parameter region with the inequality

$$m^2 > \frac{1}{4}\sum_{i=1}^{3}\left(\frac{\dot{\mathcal{R}}_i}{\mathcal{R}_i}\right)^2 - \frac{1}{2}\sum_{i\neq j=1}^{3}\left(\frac{\dot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\frac{\dot{\mathcal{R}}_j(t)}{\mathcal{R}_j(t)}\right) - \frac{1}{2}\sum_{i=1}^{3}\frac{\ddot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}, \tag{22}$$

inflaton has an oscillatory solution of the form

$$\eta(t) = \frac{1}{\sqrt{2\beta(t)}}\exp\left(-i\int\beta(t)\,dt\right), \tag{23}$$

where

$$\beta(t)=\sqrt{m^2+\frac{1}{4}\sum_{i=1}^{3}\left(\frac{\dot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\right)^2-\frac{1}{2}\sum_{i\neq j=1}^{3}\left(\frac{\dot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\frac{\dot{\mathcal{R}}_j(t)}{\mathcal{R}_j(t)}\right)-\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\ddot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\right)+\frac{3}{4}\left(\frac{\dot{\beta}(t)}{\beta(t)}\right)^2-\frac{3}{2}\frac{\ddot{\beta}(t)}{\beta(t)}} \tag{24}$$

To solve the semiclassical equation (18), rewrite the equation as follows:

$$\begin{aligned}\mathcal{R}_1(t)\mathcal{R}_2(t)\mathcal{R}_3(t)=&\frac{8\pi}{m_p^2}\left(\mathcal{N}+\frac{1}{2}\right)\frac{1}{2\beta}\frac{1}{\left(\frac{\dot{\mathcal{R}}_1(t)}{\mathcal{R}_1(t)}\frac{\dot{\mathcal{R}}_2(t)}{\mathcal{R}_2(t)}+\frac{\dot{\mathcal{R}}_2(t)}{\mathcal{R}_2(t)}\frac{\dot{\mathcal{R}}_3(t)}{\mathcal{R}_3(t)}+\frac{\dot{\mathcal{R}}_1(t)}{\mathcal{R}_1(t)}\frac{\dot{\mathcal{R}}_3(t)}{\mathcal{R}_3(t)}\right)}\\&\times\left\{\frac{1}{4}\sum_{i,j=1}^{3}\left(\frac{\dot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\frac{\dot{\mathcal{R}}_j(t)}{\mathcal{R}_j(t)}\right)+\frac{3}{4}\sum_{i=1}^{3}\left(\frac{\dot{\mathcal{R}}_i(t)}{\mathcal{R}_i(t)}\right)\frac{\dot{\beta}}{\beta}\right.\\&\left.+\frac{1}{4}\left(\frac{\dot{\beta}}{\beta}\right)^2+\beta^2+m^2\right\}\end{aligned} \tag{25}$$

The above equation can be solved perturbatively. Starting from the approximation $\mathcal{R}_{10}(t)=\mathcal{R}_{10}t^{2/3}$, $\mathcal{R}_{20}(t)=\mathcal{R}_{20}t^{2/3}$, $\mathcal{R}_{30}(t)=\mathcal{R}_{30}t^{2/3}$, and $\beta_0(t)=m$, we obtain the next order perturbative solution for $\mathcal{R}_1$

$$\mathcal{R}_{11}(t)=\frac{6\pi}{m_p^2\mathcal{R}_{20}\mathcal{R}_{30}}\left(\mathcal{N}+\frac{1}{2}\right)\left[1+\frac{1}{2m^2t^2}\right]mt^{2/3} \tag{26}$$

Similarly the next order perturbation solution for $\mathcal{R}_2$ and $\mathcal{R}_3$ are respectivly obtained as

$$\mathcal{R}_{21}(t)=\frac{6\pi}{m_p^2\mathcal{R}_{10}\mathcal{R}_{30}}\left(\mathcal{N}+\frac{1}{2}\right)\left[1+\frac{1}{2m^2t^2}\right]mt^{2/3}, \tag{27}$$

and

$$\mathcal{R}_{31}(t)=\frac{6\pi}{m_p^2\mathcal{R}_{10}\mathcal{R}_{20}}\left(\mathcal{N}+\frac{1}{2}\right)\left[1+\frac{1}{2m^2t^2}\right]mt^{2/3}, \tag{28}$$

where $\mathcal{R}_{11}$ means the next order perturbation solution for the scale factor $\mathcal{R}_1$ in the x direction and the same hold for $\mathcal{R}_{21}$ and $\mathcal{R}_{31}$, respectively in the y and z directions.

From the above three equations, it follows that

$$\begin{aligned}\mathcal{R}_{11}&\sim t^{2/3},\\ \mathcal{R}_{21}&\sim t^{2/3},\end{aligned} \tag{29}$$

and

$$\mathcal{R}_{31}\sim t^{2/3}.$$

Which shows, irrespective of direction, all scale factors follow the same power-law of expansion.

4. CONCLUSIONS

In this paper, we have studied a homogeneous and massive scalar field minimally coupled to the gravity, in the Bianchi type-I universe in the frame work of semiclassical theory of gravity. The approximate leading solution to the semiclassical Einstein, in the oscillatory phase of the inflaton after inflation, is found. The next order solution for each scale factor in their respective direction, shows that each scale factor in each direction follows $t^{2/3}$ power-law of expansion. Further, the solution for one of the scale factors dependent on the initial value of the other scale factors of other directions. Therefore it can be concluded that, evolution scale factors are mutually correlated. When $\mathcal{R}_1(t) = \mathcal{R}_2(t) = \mathcal{R}_3(t) = \mathcal{R}(t)$, the corresponding solution becomes as that of isotropic model case and the result is consistent with the result obtained in Kim and Kim (1998). From anisotropic to isotropic transition a damping mechanism is required. One of the efficient damping mechanisms, can be due to the particle creation in anisotropic models, as discussed by Hu and Parker (1978). Therefore particle production mechanism can bring isotropy in the Bianchi type-I model. The present study can account for the power-law of expansion of the scale factors in the Bianchi type-I universe, for a homogeneous and massive scalar field minimally coupled to the gravity in the frame work of semiclassical theory of gravity.

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